On the role of nonlinearities in the Boussinesq-type wave equations

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Abstract

Longitudinal mechanical waves in biomembranes are described by a Boussinesq-type wave equation. It is shown that in this case the nonlinearities are of a different type compared with conventional models of solids. The governing equation analysed in this paper is the improved Heimburg-Jackson model with two dispersive terms. The soliton-type solutions of such a wave equation are found and analysed. The existence of solitons depends on the ratio of nonlinear terms and the width of solitons is governed by dispersive terms.

Keywords: Boussinesq-type equations, Biomembrane, Nonlinearities, Dispersion

1. Introduction

The Boussinesq approximation for water waves is known from the 19th century (see Bois [1]) and is nowadays generalised also for waves in solids [2, 3]. In general terms, a Boussinesq-type model for modelling waves is based on the classical second-order wave operator but more effects are included. Altogether, it is characterised by the following effects [2]: (i) bi-directionality of waves (due to the second-order wave operator); (ii) nonlinearity (of any order); (iii) dispersion (of any order modelled by space and/or time derivatives of the fourth order at least). Such a model may be described by the following equation [2]:

$$u_{tt} - c_0^2 u_{xx} - \left(\frac{dF(u)}{du}\right)_{xx} = (\beta_1 u_{tt} - \beta_2 u_{xx})_{xx}, \tag{1}$$

where $F(u)$ is a polynomial, starting with second degree, c_0 is the velocity and β_1, β_2 are coefficients characterising dispersion. The crucial point for grasping the physical effects is certainly the structure of the nonlinear term in Eq. (1) and the signs of coefficients β_1, β_2 or the combination of possible other higher-order terms.

Many studies are devoted to the dispersion analysis of Eq. (1), i.e., the influence of the structure of its r.h.s. [2, 4, 5, 6] on dispersion relations and possible stabilities or instabilities of speeds over the large range of frequencies. Less attention is paid to the influence of nonlinearities on wave motion modelled by Eq. (1) or alike. Here the main issue is whether the nonlinear terms of $f(u)$-type or $g(u_x)$-type appear in the governing equation. Further we represent the governing equations either in terms of u (which is displacement) or in terms of $v = u_x$ (which is deformation). As shown further in Section 2, both cases have been described for modelling waves in various studies. However, the problem is not related to the formulation of governing equations only but also to the formulation of initial or boundary conditions, i.e., to the excitation of wave processes.

In this paper we start in Section 2 with the presentation of various models and then proceed to one of the crucial problems in solitonics – the existence of solitons (Section 3). Then in Section 4 we proceed to the analysis of the improved Heimburg-Jackson model and demonstrate how the solution depends on the shape of the 'pseudo-potential'. This explains the role of the $f(u)$-type nonlinearities in the model. It is also shown how the width of the soliton depends on dispersive effects. Finally, in Section 5, the final remarks are given.

2. Boussinesq-type models

In what follows, the main cases of Boussinesq-type models used for describing waves in solids are presented. For microstructured solids the governing equation for longitudinal waves in terms of displacement u is [3, 7]

$$u_{tt} - (b + \mu u_x)u_{xx} = \delta(\beta u_{tt} - \gamma u_{xx})_{xx}, \tag{2}$$

where nonlinearity of the macrostructure is taken into account, b, μ, β, γ are physical coefficients and δ is a scale factor. In terms of deformation $v = u_x$, Eq. (2) takes the form

$$v_{tt} - bv_{xx} - \frac{1}{2}\mu\left(v^2\right)_{xx} = \delta(\beta v_{tt} - \gamma v_{xx})_{xx}. \tag{3}$$

The analysis of .(2) and (3) is given by Tamm [8], Peets [9] and Berezovski et al. [10]. For longitudinal waves in rods, the governing equation in terms of deformation is derived by Porubov [11]. In original notations this equation is

$$v_{tt} - av_{xx} - c_1 \left(v^2\right)_{xx} = -\alpha_3 v_{xxtt} + \alpha_4 v_{xxxx} \tag{4}$$

with a, c_1, α_3, α_4 denoting the physical coefficients including the radius of the rod. Here $v = u_x$. Compared with Eq. (3), where dispersion effects appear due to the presence of the microstructure, the dispersion effects in Eq. (4) are due to the geometry of the rod.

In mathematical terms, dispersive effects in Eqs (1)-(3) are described by the higher-order space and space time derivatives. The structure of Eq. (2) demonstrates clearly the influence of the inertia of the microstructure and its elasticity [3] while in the case of geometrical dispersion the mixed derivative appears due to the Love assumption linking the transverse displacement w to the longitudinal deformation u_x: $w = -r\nu u_x$, where r is the radius of the rod and ν – Poisson's ratio [11].

It is possible that the dispersive effects are described by different assumptions. Bogdanov and Zakharov [6] have used the following form (in original notations)

$$\frac{3}{4}\alpha^2 v_{tt} - \beta v_{xx} + \frac{3}{2}\left(v^2\right)_{xx} = -\frac{1}{4}v_{xxxx} \tag{5}$$

in order to study long-wave and short-wave instabilities. Here α and β are the physical coefficients.

Christou and Papanicolau [12] have studied even higher-order dispersion modelled by

$$v_{tt} - \gamma^2 v_{xx} - \alpha_1 \left(v^2\right)_{xx} = \beta_1 v_{4x} + \delta_1 v_{6x}, \tag{6}$$

where γ, α_1, β_1, δ_1 are coefficients with $\delta_1 > 0$. Maugin [13] has introduced the Maxwell-Rayleigh equation

$$u_{tt} - u_{xx} - \left(\frac{dF(u)}{du}\right)_{xx} = \gamma(u_{xx} - u_{tt})_{tt}, \tag{7}$$

where dispersive effects are influenced by the fourth-order time derivatives.

The nonlinearities may be modelled also differently. For waves in biomembranes, Heimburg and Jackson [14] have assumed that the sound velocity in the membrane depends on the density changes $\Delta\rho^A = u$:

$$c^2 = c_0^2 + pu + qu^2 + \ldots \quad , \tag{8}$$

where c_0 is the velocity in the unperturbed state, $p < 0$ and $q > 0$ are coefficients determined from the experiments [14]. Then the longitudinal waves are described by the governing equation

$$u_{tt} - \left[\left(c_0^2 + pu + qu^2\right) u_x\right]_x = -h_1 u_{xxxx}, \tag{9}$$

where h_1 characterises the strength of dispersion described by an added *ad hoc* term (u_{xxxx}).

Engelbrecht et al., [15] and Tamm and Peets [16] have derived an improved version of Eq. (9) eliminating the possible instability of the solution due to infinite velocity at high frequencies. This equation is

$$u_{tt} - \left[\left(c_0^2 + pu + qu^2\right) u_x\right]_x = -h_1 u_{xxxx} + h_2 u_{xxtt}, \tag{10}$$

where h_2 is an additional physical coefficient. With such dispersive terms in Eq. (10), the internal structure of the biomembrane is better described than in the original Eq. (9) [15].

In order to compare Eq. (10) with other Boussinesq-type models shown above, we rewrite it like

$$u_{tt} - \left(c_0^2 + pu + qu^2\right) u_{xx} - (p + 2qu_x)u_x^2 = -h_1 u_{xxxx} + h_2 u_{xxtt}. \tag{11}$$

It is clearly seen that nonlinearities in Eq. (11) differ considerably from other models (Eqs (1) – (7)) – usually the nonlinearities in Boussinesq-type equations are in terms of u_x, however in Eq. (11) the nonlinearities are in terms of plain u, which is unusual for a Boussinesq-type models [2]. It must also be stressed that the nonlinearities in Eq. (11) can be expressed as $[(pu + qu^2)\, u_x]_x = p(u^2)_{xx}/2 + q(u^3)_{xx}/3$, which in principle is different from models (5), (6) and alike. Namely, models (5), (6) include $(v^2)_{xx}$ and $(v^3)_{xx}$ as these nonlinearities are in terms of deformation $v = u_x$.

3. Soliton solutions to Boussinesq-type models

The standard way to find soliton solutions is to seek for stationary solutions as functions of a moving coordinate $\xi = x - ct$, where c is the speed of the soliton. For Eq. (1), named also as 'Boussinesq Paradigm Equation' [2], the soliton solution for $F(u) = 1/3\alpha u^3$ is

$$u = A\,\mathrm{sech}^2\left[b(x - ct)\right], \tag{12a}$$

$$A = 3\frac{c_0^2 - c^2}{2\alpha}, \quad b = \frac{1}{2}\sqrt{\frac{c_0^2 - c^2}{\beta_2 - \beta_1 c^2}}, \tag{12b}$$

where c is a speed of a soliton. It is important to stress the existence of sub-critical solitons with negative amplitudes and super-critical solitons with positive amplitudes. The general condition for existence of such solitons is $\beta_2 < \beta_1 c_0^2$; sub-critical solitons have $c < c_0\sqrt{\beta_2/\beta_1}$ and super-critical solitons obey $|c| > c_0$.

Porubov [17] has derived the same solution (12a) with A and b in terms of the rod radius r and Poisson's ratio ν. Janno and Engelbrecht [18] have found the existence of a single solitary wave for Eq. (3) provided the conditions

$$\frac{c^2-b}{\beta c^2-\gamma} > 0, \tag{13a}$$

$$\mu \neq 0, \quad \beta c^2-\gamma \neq 0, \quad c^2-b \neq 0 \tag{13b}$$

are satisfied. Then the solitary wave is

$$v(x-ct) = A\,\mathrm{sech}^2\left[\frac{1}{2}\kappa(x-ct)\right], \tag{14a}$$

$$A = 3(c^2-b)/\mu, \quad \kappa = \sqrt{\frac{c^2-b}{\delta(\beta c^2-\gamma)}}. \tag{14b}$$

It is easy to check that the solutions (12) and (14) coincide. If the microstructure is also considered nonlinear then the structure of (14) is modified [18] and the outcome is a deformed soliton [18].

The solitons described by (12) and (14) are in terms of v, i.e., in terms of deformation. We proceed now to Eqs (9)–(11) in terms of density changes u which is proportional to the displacement along x-axis. There is a significant difference between the solitons or solitary waves explained in terms of displacements or deformation. Namely, a single pulse (sech^2-type) of a displacement means actually a sign-changing deformation (Fig. 1a), while a single pulse (sech^2-type) of a deformation means the change of the displacement from one level to another (Fig. 1b).

4. Analysis of the improved Heimburg-Jackson model

For convenience of the following analysis Eq. (10) is rewritten in a dimensionless form:

$$U_{TT} = [(1+PU+QU^2)U_X]_X - H_1U_{XXXX} + H_2U_{XXTT}, \tag{15}$$

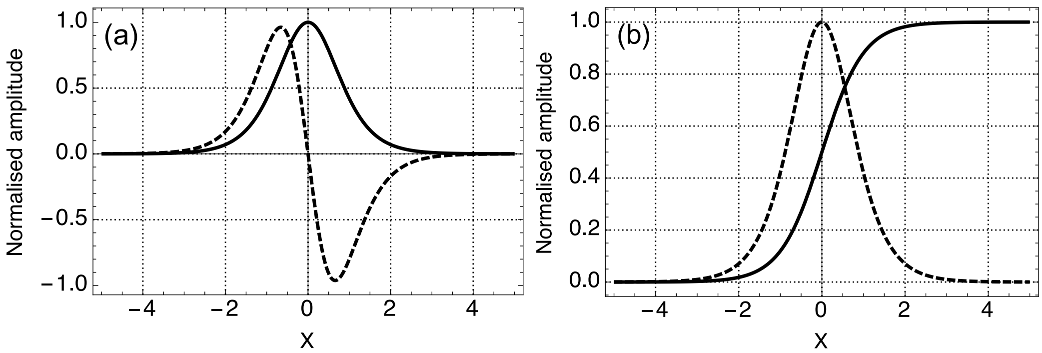

Figure 1: Left: sech^2-type displacement U (solid line) and deformation U_X (dashed line), right: sech^2-type deformation U_X (dashed line) and displacement U (solid line).

with $X = x/l$, $T = c_0t/l$, $U = u/\rho_0$ and $P = p\rho_0/c_0^2$, $Q = q\rho_0^2/c_0^2$, $H_1 = h_1/(c_0^2l^2)$, $H_2 = h_2/l^2$; where l is a certain length [15].

Solutions for Eq. (15) are found by seeking waves of permanent shape and size by looking for solutions such that

$$V = V(\xi), \quad \xi = X - \beta T, \tag{16}$$

where is V is some function and β is constant wave velocity [19, 20]. Substituting this into Eq. (15) we get

$$\beta^2 V'' = ((1 + PV + QV^2)V')' - H_1 V'''' + H_2\beta^2 V''''. \tag{17}$$

Integrating Eq. (17) twice we get after some rearranging

$$(H_1 - H_2\beta^2)V'' = (1 - \beta^2)V + \frac{1}{2}PV^2 + \frac{1}{3}QV^3 + AV + B, \tag{18}$$

where A and B are constants of integration. Since we are looking for solitary wave, then we may add boundary conditions that $V, V', V'' \to 0$ as $X \to \pm\infty$ and therefore $A, B = 0$ [19, 20]. Now the Eq. (18) is multiplied by V' and integrated to get

$$(H_1 - H_2\beta^2)(V')^2 = (1 - \beta^2)V^2 + \frac{1}{3}PV^3 + \frac{1}{6}QV^4, \tag{19}$$

which can be rewritten as

$$(H_1 - H_2\beta^2)(V')^2 = \Phi_{eff}(V), \tag{20}$$

where

$$\Phi_{eff}(V) = (1-\beta^2)V^2 + \frac{1}{3}PV^3 + \frac{1}{6}QV^4 \tag{21}$$

is a fourth-order 'pseudo-potential' which allows us to investigate the effect of the parameters P, Q and β on the solutions. Note that for the classical KdV equation the 'pseudo-potential' is of the third order [20].

In case of $H_2 = 0$ the 'pseudo-potential' (21) also applies for the Heimburg-Jackson model (9) and has been analysed by Lautrup et al. [21] for a particular set of parameters that are relevant for the solitary wave propagation in biomembranes ($P < 0$, $Q > 0$). Mathematically, however, the parameters P, Q can have wider range of values (positive or negative) enlarging so the class of possible solutions of model (15). General mathematical analysis of Eq. (21) (with $H_2 = 0$) has been given by Freistühler and Höwing [22]. Here we present an analysis based on understandings of wave mechanics in the context of the Boussinesq's paradigm [2, 3, 4].

The four zeros of the polynomial (21) are

$$V_{1,2} = 0 \quad \text{and} \quad V_{3,4} = \frac{P}{Q}\left(-1 \pm \sqrt{1 - \frac{(1-\beta^2)6Q}{P^2}}\right). \tag{22}$$

Double zero at $V_{1,2} = 0$ indicates the saddle point, which is minimal requirement for the existence of solitary waves [19, 20]. It can also be deduced from Eqs (21) and (22) that the additional condition for the velocity β is

$$1 > |\beta| > \sqrt{1 - \frac{P^2}{6Q}}, \tag{23}$$

which is also the case for the Heimburg-Jackson model (9) [21]. The analytical solution of Eq. (11) is

$$u(\xi) = \frac{6(\beta^2-1)}{P(1+\sqrt{1+6Q(\beta^2-1)/P^2}\cosh(\xi\sqrt{(1-\beta^2)/(H_1 - H_2\beta^2)})}, \tag{24}$$

where $\xi = X - \beta T$ and β is the velocity of the solitary wave. We also note that if $H_2 = 0$ then the Eq. (24) reduces to the solution of Eq. (9) which can be found in [21].

Four typical cases of 'pseudo-potential' $\Phi_{eff}(V) > 0$ and corresponding wave profiles are depicted in Figs 2 and 3 respectively. Solitary wave solutions are possible when $\Phi_{eff}(V)$ has a local minimum at $V = 0$ with at least one local maximum next to it. This means that in our case solitary wave solutions are possible in the region where $\Phi_{eff}(V) > 0$. It can be seen in Figs 2 and 3 that positive amplitude solitary waves emerge when $P < 0$ and negative amplitude solitary waves when $P > 0$. The analysis of Eq. (21) shows that

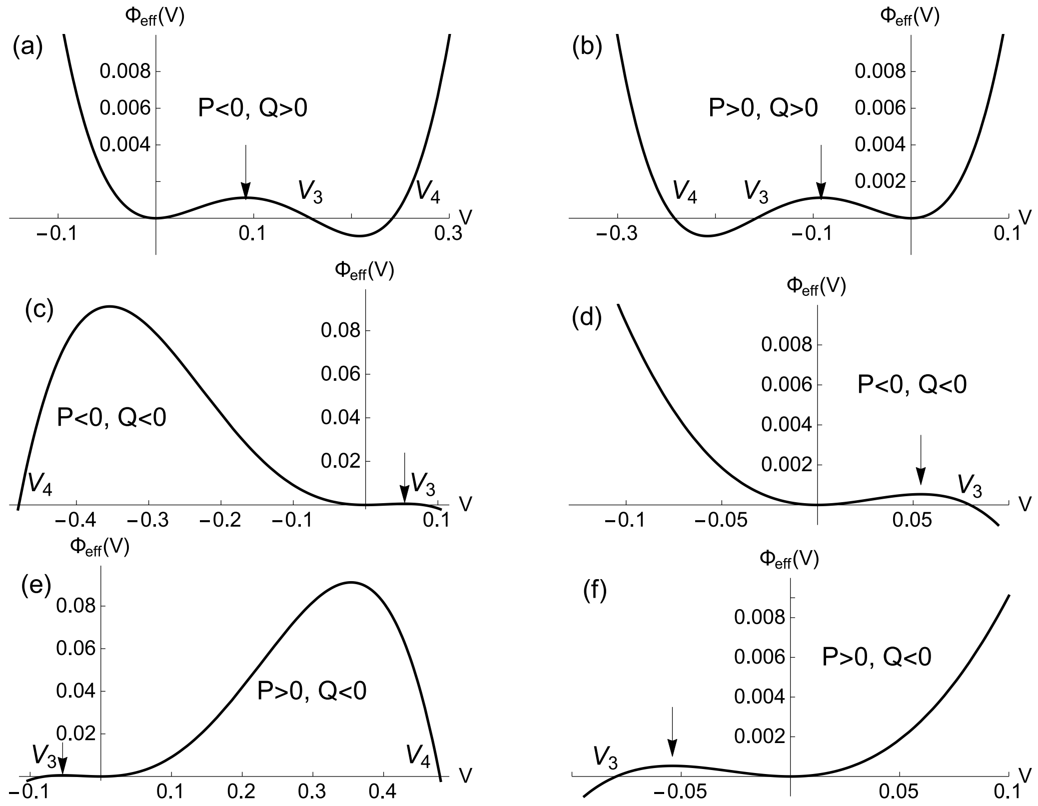

Figure 2: Shape of the effective potential (21) for four different cases. $\beta = 0.7$, $|P| = 16$ and $|Q| = 80$. Subfigures (d) and (f) represent blowups of the effective potential (21) near V_3 for subplots (c) and (e) respectively. Arrows point to the local maxima near V_3.

for a given set of parameters the solitons amplitude is equal to

$$V_3 = \frac{P}{Q}\left(-1 + \sqrt{1 - \frac{(1-\beta^2)6Q}{P^2}}\right). \tag{25}$$

This is because V_3 is always closer to the saddle point than V_4 (see Fig. 2 and corresponding wave profiles in Fig. 3). In case of two adjacent maxima to the saddle point at $V = 0$, the local maximum closer to the saddle point is realised (c.f. Figs 2 c,e).

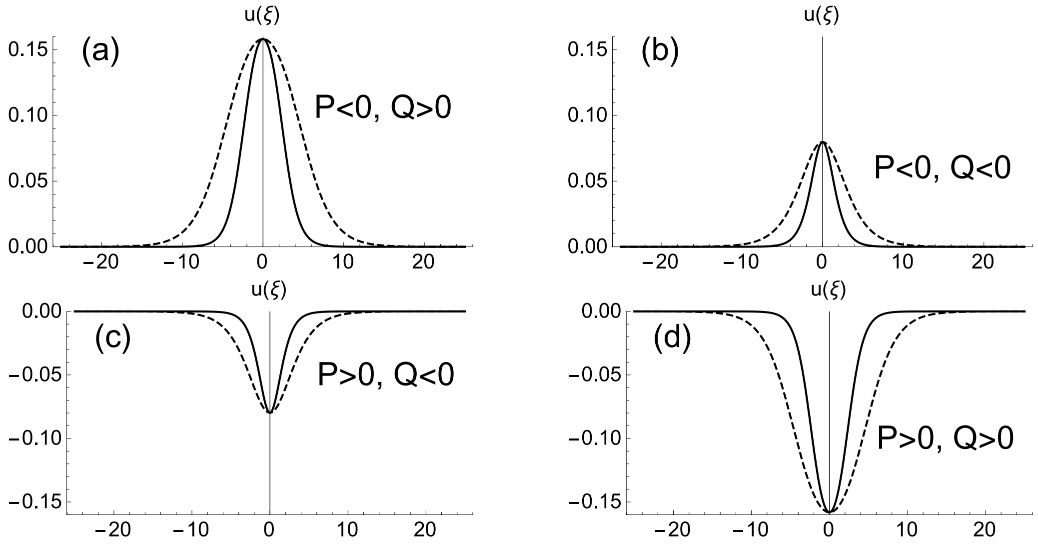

Figure 3: Solitary wave solutions of Eq. (10) (solid line) and Eq. (9) (dashed line) in case of $\beta = 0.7$, $|P| = 16$,$|Q| = 80$, $H_1 = 2$ and $H_2 = 3$.

Equation (25) also sets constraints to parameters P and Q – solitary wave solutions only exist when $Q < P^2/((1-\beta^2)6)$, i.e., parameter Q has an upper bound but no lower bound.

It is also interesting to consider the extreme cases of $P = 0$ and $Q = 0$. When $Q = 0$ the 'pseudo-potential' (21) becomes the third order polynomial and has double zero at $V = 0$ and one adjacent to it, which is similar behaviour as in case of the KdV equation [20]. In case of $P = 0$ the 'pseudo-potential' is symmetrical with respect to the vertical axis with double zero at $V = 0$.

In order to demonstrate the effect of the dispersion parameter H_2 on the analytical solution of Eq. (24), a case with $H_2 = 0$ is also plotted in Fig. 3 (dashed line) which is a solution for the original Heimburg-Jackson model (9). It can be seen in Fig. 3 that the second dispersion parameter H_2 makes solution (24) more localised. The effect is stronger in case of anomalous dispersion ($H_2 > H_1$, see [24] for details) than in case of the normal dispersion ($H_2 > H_1$).

5. Final remarks

The longitudinal waves in biomembranes are described by a Boussinesq-type governing equation [14, 15], Leaving aside the physical description [14, 25], two comments are essential concerning the structure of the governing equation.

First, contrary to the widely used models for waves in fluids and solids [2], where terms like $u_x u_{xx}$, $(u_x)^2 u_{xx}$ appear, here the terms like uu_{xx}, $u^2 u_{xx}$ reflect the nonlinear effects. This due to the special structure of the lipids which form the biomembrane. Although only the combination $P < 0$, $Q > 0$ corresponds to the case of biomembranes [14, 21], the other cases of P and Q demonstrate effectively the role of nonlinearities in the governing equation (15). It must be noted that in principle relation (8) may involve even higher order nonlinear terms like ru^3 [26].

Second, it has been proposed that the original Heimburg-Jackson model (9) [14] involving one dispersive term (u_{xxxx}) must be improved involving two dispersive terms (u_{xxxx}, u_{xxtt}) [15]. The improved model is based on concepts of continuum mechanics where the term u_{xxtt} actually describes the influence of the inertia of the microstructure. It has been shown that in this case the possible instability at higher frequencies is eliminated.

In the present paper the existence of solitons is analysed based on the improved wave equation (10). The soliton solution (24) derived by conventional technique in term of $\xi(X - \beta T)$ includes the coefficients of nonlinear terms and the coefficients of dispersive terms (cf. [14, 23]). The conditions for the existence of such a solution are presented. It has been shown that in general terms the soliton (24) may be narrower than the case without the term u_{xxtt} (see Fig. 3). This is a direct reflection that the internal structure of a biomembrane has inertia.

It must be stressed that the governing equation (10) describes the longitudinal wave with an amplitude u. From the theory of rods in continuum mechanics it is shown that the transverse displacement is then proportional to u_x. In case of a soliton-type wave u_x has a typical shape of a multivalued profile (see Fig. 1a). This coincides qualitatively with measurements by Tasaki [27, 28] who measured the transverse displacement of the nerve fibre which is similar to the dashed line U_X in Fig. 1a.

We note that the emergence of solitons modelled by Eq. (15) from an initial input is analysed in [16, 24]. The analysis of the collision of solitons should show whether the obtained solitons are solitons in the classical sense (interaction without radiation). The fascinating problem of wave propagation in biomembranes needs further studies. Definitely the dissipative effects and coupling with action potential should be analysed in detail (see for example [29]).

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